1. Semisimplicity and Schur's lemma - 08/12/09

A representation $\rho : G \to \mathrm{Aut}(V)$ of a finite group is said to be *irreducible* if there is no nontrivial vector subspace $U \subset V$ which is left stable under the action of G.

If there is an inner-product lurking in the background, and ρ is a unitary representation, then for every $U \subseteq V$ there exists a unique $U^\perp$ such that $V = U \oplus U^\perp$. If U is such that:

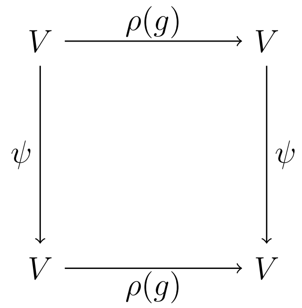
$$\rho(g).u \in U \text{ for all } u \in U \text{ and for all } g \in G$$

Then it follows that $U^\perp$ is such that:

$$\rho(g)^*.u \in U \text{ for all } u \in U \text{ and for all } g \in G$$

But since the representation is unitary, $\rho(g)^* = \rho(g)^{-1}$, thus every unitary representation of G decomposes into an orthogonal direct sum of irreducible ones. This is the definition of *semisimplicity*.

Now fix some representation $\rho : G \to V$. If ρ is irreducible, then the only maps ψ with the property that: for all $g \in G$ the following diagram commutes:



are the scalar multiples of the identity.

To see this not that if λ is an eigenvalue of ψ then the eigenspace of ψ corresponding to λ is stable under all of G and must thus be the whole space. Over $\mathbb{C}$ every linear operator has at least one eigenvalue.

More generally, suppose that V decomposes as:

$$V = U \oplus U^{\perp}$$

with each of U and $U^{\perp}$ irreducible. If the representation of G restricted to U is *not* isomorphic to the representation of G on $U^{\perp}$, then the the only $\psi : V \to V$ which commtes with the action of G are those of the form:

$$\psi = \lambda I_U \oplus \mu I_{U^{\perp}}$$

where I_U denotes the identity on U and $I_{U^{\perp}}$ denotes the identity of $U^{\perp}$.

If they *are* isomorphic, then we can write:

$$V = U \otimes W$$

where W is a *multiplicity space* of dimension 2. In this basis the representation looks like:

$$\rho(g) = \rho_U(g) \otimes I_W$$

where ρ_1 is the restriction of ρ to U and I_W is the identity matrix on W.

The ψ which commute with the action of G are now of the form:

$$\psi = \lambda I \otimes A$$

where I denotes the identity matrix acting on U, and A is *any* matrix acting on W.

Finally if $\rho : G \to V$ is an arbitrary representation, and the underlying vector space decomposes as

$$V = \bigoplus_k \big(U_k \otimes W_k\big)$$

with the U_k denoting pairwise non-isomorphic irreducible representations, and the W_k their multiplicity spaces, then the representation looks like:

$$\rho(g) = \bigoplus_k \rho_k(g) \otimes I_{W_k}$$

and the $\psi : V \to V$ which commute with the action of G are those of the form:

$$\psi = \bigoplus_k \lambda_k I_{U_k} \otimes A_k$$

This is more or less *Schur's lemma.*