1. Permutation Groups 21/11/09

A permutation group is a homomorphism from a finite group G into the symmetric group $\mathfrak{S}_n$ for some n. We prefer to think of the symmetric group $\mathfrak{S}_n$ acting *on the right* of some n element set Ω. For example if we take $\Omega = \{1, 2, 3\}$ then:

$$1^{(132)} = 3$$
$$2^{(132)} = 1$$
$$3^{(132)} = 2$$

To describe such a homomorphism it suffices to give explicitely the action of G on some n-element set Ω and verify that it satisfies the laws of a *group action*. That is to say:

$$(x^{g_1})^{g_2} = x^{(g_1 g_2)}$$
$$x^1 = x$$

Every group acts on itself by right multiplication. If $|G| = n$ then we we have a homomorphism from G into $\mathfrak{S}_n$. To describe this action explicitely we let $\Omega = G$, Now:

$$x^g = xg$$

and this clearly satisfies the laws of a group action.

Similarly, every group acts on itself by left multiplication. Again let $\Omega = G$. Now:

$$x^g = g^{-1}x$$

Pay special attention to the role of the inverse when acting on the left. To see that this truly satisfies the laws of a group action note that:

$$\begin{aligned}(x^{g_1})^{g_2} &= (g_1^{-1}x)^{g_2}\\ &= g_2^{-1}g_1^{-1}x\\ &= (g_2 g_1)^{-1}x\\ &= x^{(g_1 g_2)}\end{aligned}$$

When we have several different groups acting on the same set, it is convenient to introduce some additional notation, to this end we introduce the following formal definition.

Definition 1.1 (Permutation group)**.** *A* permutation group *is a triple* (G, Ω, ρ) *where* G *is a finite group,* Ω *is a finite set and* ρ *is a homomorphism:*

$$\rho : G \to \mathrm{Aut}(\Omega).$$

Of course if $|\Omega| = n$ then $\mathrm{Aut}(\Omega) = \mathfrak{S}_n$. We write $\alpha^{\rho(g)}$ to indicate the action of a permutation $g \in G$ on a point $\alpha \in \Omega$.

There is a sense in which the left action and the right action of a group on itself are *equivalent*. To make this sense precise we introduce a second definition.

Definition 1.2 (Permutation Equivalence)**.** *Two permutation groups* (G_1, Ω_1) *and* (G_2, Ω_2) *are said to be* permutation equivalent *if there is an isomorphism* $\varphi : G_1 \to G_2$ *and a bijection* $\eta : \Omega_1 \to \Omega_2$ *such that for every* $g \in G_1$ *the following diagram commutes:*

$$\begin{array}{ccc} \Omega_2 & \xrightarrow{\quad g \quad} & \Omega_2 \\ \Big\downarrow \eta & & \Big\downarrow \eta \\ \Omega_1 & \xrightarrow[\varphi(g)]{\qquad} & \Omega_1 \end{array}$$

We can see now by taking:

$$\begin{aligned} \eta : G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

and φ to be the identity that the left and right actions of a group on itself are equivalent:

$$\begin{aligned} \eta(x)^{\rho_2(\varphi(g))} &= (x^{-1})^{\rho_2(g)} \\ &= g^{-1}x^{-1} \\ &= (xg)^{-1} \\ &= \eta(xg) \\ &= \eta(x^{\rho_1(g)}) \end{aligned}$$

Note that η is most definitely *not* an isomorphism of groups, it is merely a bijection form the *set* G to itself.

As a second example of a permutation group let G be any finite group, and let H be any subgroup. Let $\{x_0, \dots, x_{r-1}\}$ be a traversal of the cosets of H in G, with $x_0 = 1$. Let:

$$\Omega = \{H, Hx_1, \dots Hx_{r-1}\}$$

That is Ω is the set of right cosets of h in G. G acts naturally on the right of Ω:

$$(Hx_i)^g = Hx_i g = Hx_j$$

where x_j is such that there exists a $h \in H$ with $x_j = hx_i g$.

Alternatively we could have taken the *left* coset of H in G:

$$\Omega' = \{H, x_1^{-1}H, \dots, x_{r-1}^{-1}H\}$$

Now G acts on the *left* of Ω':

$$(x_i^{-1}H)^g = g^{-1}x_i^{-1}H = x_j^{-1}H$$

with x_j exactly as in the previous example.

Note the appearance of the inverse once again when acting on the left. In fact, these two actions are equivalent as can be seen by taking:

$$\eta : \Omega \to \Omega'$$
$$Hx_i \mapsto x_i^{-1}H$$

and $\varphi : G \to G$ to be the identity.

We shall see in the next post, that all *transitive* permutation groups are of this form, and all permutation groups are *disjoint orbits* of the transitive ones.