1. Matrices - 06/11/09

The purpose of this post is to explain matrix multiplication, the trace operator and the transpose operator in terms of duality and tensor products.

Let E be a vector space of finite dimension n and let E^* denote its dual. We are interested in the two vector spaces: $L(E) = E \otimes E^*$ and $L(E^*) = E^* \otimes E$. On account of the duality we have a map:

$$\text{ev} : E^* \otimes E \to \mathbb{C}$$

with the property that for any basis $\{e_1, \dots, e_n\}$ of E there exists a unique dual basis $\{e^1, \dots, e^n\}$ of E^* such that:

$$\text{ev}[e^j \otimes e_i] = \begin{cases} 1 & \text{if i = j} \\ 0 & \text{otherwise} \end{cases}$$

We also have a map:

$$\begin{aligned} \text{ev}^* : E^* \otimes E &\to \mathbb{C} \\ e^j \otimes e_i &\mapsto \text{ev}(e_i \otimes e^j) \end{aligned}$$

As vector spaces $L(E) = E \otimes E^*$ and $L(E^*) = E^* \otimes E$ are naturally isomorphic:

$$\begin{aligned} T : E \otimes E^* &\to E^* \otimes E \\ e_i \otimes e^j &\mapsto e^j \otimes e_i \end{aligned}$$

Now, given a matrix:

$$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

We can either think of it as a vector in $L(E) = E \otimes E^*$ and write it as:

$$\boxed{A = a^i{}_j \; e_i \otimes e^j}$$

or we can think of it as a vector in $L(E^*) = E^* \otimes E$ and write it as:

$$\boxed{A = a_j{}^i \; e^j \otimes e_i}$$

Here we are making use of the physicists summation convention. In the first interpretation the usual trace operator is given by ev^* while in the second it is given by ev.

It is possible to define a multiplication on $L(E) = E \otimes E^*$ via:

$$A.B = (1 \otimes \mathrm{ev} \otimes 1)[A \otimes B]$$

Similarly, it is also possible to define a multiplication on $L(E^*) = E^* \otimes E$ via:

$$A.B = (1 \otimes \mathrm{ev}^* \otimes 1)A \otimes B$$

Making use of the notation $\boxed{E_i^{\ j} = e_i \otimes e^j}$, we have:

$$\begin{aligned} E_i^{\ j}.E_k^{\ l} &= (1 \otimes \mathrm{ev} \otimes 1)[e_i \otimes e^j \otimes e_k \otimes e^l] \\ &= \mathrm{ev}[e^j \otimes e_k] e_i \otimes e^l \\ &= \begin{cases} E_i^{\ l} & \text{if j = k} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Similarly, making use of the notation $\boxed{E^i_{\ j} = e^j \otimes e_i}$, we have:

$$\begin{aligned} E^j_{\ i}.E^l_{\ k} &= (1 \otimes \mathrm{ev}^* \otimes 1)[e^j \otimes e_i \otimes e^l \otimes e_k] \\ &= \mathrm{ev}^*[e_i \otimes e^l] e^j \otimes e_k \\ &= \begin{cases} E^j_{\ k} & \text{if i = l} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

From this we observe the following:

$$\boxed{T(A.B) = T(B).T(A)}$$

In conclusion, a matrix may be thought of as either as an element of $L(E) = E \otimes E^*$ or as an element from $E^* \otimes E = L(E^*)$. It is a question of interpretation. The spaces $L(E)$ and $L(E^*)$ are isomorphic as vector spaces but *anti*-isomorphic as groups.