1. Group Rings and Regular Representations

For any finite group G the *group ring* $\mathbb{C}G$ of G is the group of formal linear combinations of elements of G, with coefficients in $\mathbb{C}$ and multiplication defined on the basis elements and then extended linearly:

$$\Big(\sum_{g\in G} a_g\ g\Big)\Big(\sum_{g\in G} b_g\ g\Big) = \sum_g\sum_h (a_g b_h)\ gh = \sum_g\Big(\sum_h a_{gh^{-1}}b_h\Big)g$$

Even though it is called a group *ring* this important object actually has the structure of a $\mathbb{C}$-*algebra*. That is it is both a ring, and a finite dimensional vector space over $\mathbb{C}$. In reading what follows it is important to keep clear in your mind when we are treating $A = \mathbb{C}G$ as a ring, and when we are treating it as a vector space over $\mathbb{C}$.

A representation of of a a $\mathbb{C}$-algebra A on a finite dimensional vector space V is an algebra homomorphism:

$$\rho : A \to \mathrm{End}(V)$$

Here $\mathrm{End}(V) = V^* \otimes V$ is the space of endomorphisms of V. If V is of dimension n then once a basis has been chosen $\mathrm{End}(V)$ may be thought of as the algebra of n by n matrices.

Every representation ρ of a group G on a vector space V lifts naturally to a representation $\hat{\rho}$ of the algebra $A = \mathbb{C}G$ on the same vector space.

$$\hat{\rho}\left(\sum_g a_g g\right) = \sum_g a_g \rho(g)$$

Since these two things are really "the same" in a certain category theoretical sense which I don't want to go into just now (I'm starting to sound like a physicist, aren't I?), we usually drop the hat and swap back and forth freely between the representation of the group, and the representation of the associated group ring.

Now, there are two natural representations of G on the group ring $\mathbb{C}(G)$. Right multiplication:

$$R : G \to \mathrm{Aut}(A)$$

given by:

$$\begin{aligned} R(x).\left(\sum_g a_g\ g\right) &= \sum_g a_g\ gx \\ &= \sum_g a_{gx^{-1}}\ g \end{aligned}$$

and left multiplication by the inverse:

$$L : G \to \mathrm{Aut}(A)$$

given by:

$$\begin{aligned} L(x).\left(\sum_g a_g\ g\right) &= \sum_g a_g\ x^{-1}g \\ &= \sum_g a_{xg}\ g \end{aligned}$$

The linear map:

$$\begin{aligned} \sigma : A &\to A \\ g &\mapsto g^{-1} \end{aligned}$$

is an *intertwiner* between these two representations. In other words, upto conjugation they are really the same representation.

$$\begin{aligned} \sigma(R(x).g) &= \sigma(gx^{-1}) \\ &= xg^{-1} \\ &= L(x).\sigma(g) \end{aligned}$$

On the dual space A^* of linear functions on G we have

$$\begin{aligned} R^* : G &\to \mathrm{Aut}(A^*) \\ L^* : G &\to \mathrm{Aut}(A^*) \end{aligned}$$

given by:

$$\begin{aligned} R^*(x).f[g] &= f[gx^{-1}] \\ L^*(x).f[g] &= f[xg] \end{aligned}$$

One can check that:

$$\begin{aligned} R^*(x).f[R(x).g] &= f[g] \\ L^*(x).f[L(x).g] &= f[g] \end{aligned}$$

Since the left and right actions clearly commute, we actually have a representation:

$$L \otimes R : G \times G \to A$$

given by:

$$\begin{aligned}(L \otimes R)(x,y).\left(\sum_g a_g\ g\right) &= \sum_g a_g\ x^{-1}gy \\ &= \sum_g a_{xgy^{-1}}\ g\end{aligned}$$

Simmilarly we have:

$$L^* \otimes R^* : G \times G \to A^*$$

given by:

$$(L^* \otimes R^*)(x,y)f[z] = f[xzy^{-1}]$$

The linear map:

$$\begin{aligned}\psi : A &\to A^* \\ g &\mapsto \delta_{g^{-1}}\end{aligned}$$

is an intertwiner between these two representations of $G \times G$.

$$\begin{aligned}\psi((L \otimes R)(x,y).g)[z] &= \psi(x^{-1}gy)[z] \\ &= \delta_{x^{-1}gy}[z] \\ &= \delta_g[xzy^{-1}] \\ &= (L^* \otimes R^*)(x,y).(\psi(g)[z])\end{aligned}$$

The inverse map is given by:

$$\begin{aligned}\psi^{-1} : A^* &\to A \\ f &\mapsto \sum_g f(g)g^{-1}\end{aligned}$$

We have all the ingredients now to prove the Artin-Wedderburn theorem for semisimple rings.