# 1. Conjugacy Classes of the Symmetric Group I - 24/11/09

Recall that an *integer partition* is simply a list of non-increasing integers. For example, $\lambda = (5, 3, 3, 1, 1, 1)$ is a partition.

For any partition $\lambda$, let $m_r(\lambda)$ denote the number of parts of $\lambda$ of length $r$. In our example. $m_1(\lambda) = 3$, $m_2(\lambda) = 0$, $m_3(\lambda) = 2$ $m_4(\lambda) = 0$, $m_5(\lambda) = 1$, $m_6(\lambda) = 0$. etc...

Let us define:

$$\boxed{z_\lambda = \prod_r m_r(\lambda)! r^{m_r(\lambda)}}$$

To any permutation $\sigma$ we may associate an integer partition by reading off the lengths of the disjoint cycles. For example, the permutation $\sigma = (137)(25)(68)(9)$ has cycle type $(3, 2, 2, 1)$.

One may convince oneself that $\frac{n!}{z_\lambda}$ counts the number of permutations in $\mathfrak{S}_n$ of cycle type $\lambda$. For example

$$\begin{aligned} \frac{4!}{z_{(2,2)}} &= \frac{4!}{2!2^2} \\ &= \frac{4.3.2.1}{(2.1)(2.2)} \\ &= 3 \end{aligned}$$

Indeed there are three permutations with cycle type $(2, 2)$, namely:

$$\begin{gathered} (12)(34) \\ (13)(24) \\ (14)(23) \end{gathered}$$

Note that there are $4! = 24$ ways we could have arbitrary placed the numbers one through 4 inside the brackets to get the right "shape". However, the vast majority of these do not conform to our convention for writing down the cycle type of a permutation. For example:

$$\begin{matrix} (21)(34) & (12)(43) & (21)(43) \\ (24)(31) & (13)(42) & (31)(42) \\ (23)(41) & (14)(32) & (32)(41) \end{matrix}$$

are all wrong, because the smallest element of each cycle doesn't come first. While:

$$(34)(12)$$
$$(24)(13)$$
$$(23)(14)$$

are wrong because the cycles are not properly ordered by the size of their first elements. The remaining posibilities:

| | | |
|---|---|---|
| (34)(21) | (43)(12) | (43)(21) |
| (31)(24) | (42)(13) | (42)(31) |
| (41)(23) | (32)(14) | (41)(32) |

suffer from both types of fault. For every "correct" way of writing down the cycle type of a permutation, there are precisely $z_\lambda - 1$ "malformed" variations:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| **(12)(34)** | (34)(12) | (21)(34) | (12)(43) | (21)(43) | (34)(21) | (43)(12) | (43)(21) |
| **(13)(24)** | (24)(13) | (24)(31) | (13)(42) | (31)(42) | (31)(24) | (42)(13) | (42)(31) |
| **(14)(23)** | (23)(14) | (23)(41) | (14)(32) | (32)(41) | (41)(23) | (32)(14) | (41)(32) |

The purpose of this post is to show that two permutations $\sigma$ and $\omega$ are conjugate in the symmetric group, if and only if they have the same cycle type.

To this end we introduce the notion of a "standard" permutation of given shape. For example if $\lambda = (3, 2, 2, 1)$ then the "standard" permutation of shape $\lambda$ is:

$$s_\lambda = (123)(45)(67)(8)$$

Any permutation $\omega$ of shape $\lambda$ can be conjugated to the standard permutation $s_\lambda$. To see this, let $\omega = c_1 c_2 \cdots c_r$ where the $c_k$ are the individual cycles, and let $t_k$ denote the smallest element in the $k$th cycle.

Now, each $i \in [n]$ can be written in the form $(\lambda_1 + \cdots \lambda_{k-1} + s)$ for some $1 \le k \le r$ and some $0 \le s \le \lambda_k - 1$. Similarly, each $i \in [n]$ can be written in the form $t_k^{c_k c_k \ldots c_k}$ where the cyclic permutation $c_k$ is applied $s$ times.

Let $\pi$ be the permutation which sends $(\lambda_1 + \cdots \lambda_{k-1} + s)$ to $t_k^{c_k c_k \ldots c_k}$ ($s$-times). We have:

$$\begin{aligned}
i^{\pi\omega\pi^{-1}} &= (\lambda_1 + \cdots \lambda_{k-1} + s)^{\pi\omega\pi^{-1}} \\
&= t_k^{c_k c_k \ldots c_k \omega\pi^{-1}} \quad s \text{ times} \\
&= t_k^{c_k c_k \ldots c_k \pi^{-1}} \quad (s+1) \text{ times} \\
&= \begin{cases} (\lambda_1 + \cdots \lambda_{k-1} + s + 1) & \text{if } s+1 < \lambda_k \\ (\lambda_1 + \cdots \lambda_{k-1}) & \text{if } s+1 = \lambda_k \end{cases} \\
&= i^{s_\lambda}
\end{aligned}$$

Thus having the same cycle type is a *sufficient* condition for two permutations to be conjugate. To see that it is also a necessary condition, lets look at the eigenvalues the standard permutation of shape $\lambda$. For example, if $\lambda = (4, 3, 3, 2)$ then $s_\lambda = (1234)(567)(89A)(BC)$ (using hexadecimal).

$$\begin{pmatrix}
0 & 1 & 0 & 0 & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & & \\
0 & 0 & 0 & 1 & & & & & & & & \\
1 & 0 & 0 & 0 & & & & & & & & \\
 & & & & 0 & 1 & 0 & & & & & \\
 & & & & 0 & 0 & 1 & & & & & \\
 & & & & 1 & 0 & 0 & & & & & \\
 & & & & & & & 0 & 1 & 0 & & \\
 & & & & & & & 0 & 0 & 1 & & \\
 & & & & & & & 1 & 0 & 0 & & \\
 & & & & & & & & & & 0 & 1 \\
 & & & & & & & & & & 1 & 0
\end{pmatrix}$$

The important point, is that the standard permutation is *block diagonal.* Thus to diagonalize the matrix, we diagonalize the individual blocks separately. But the individual blocks are cyclic permutations, which we know how to diagonalize.

For our example permutation the eigenvalues, with multiplicity, are given by:

$$\{(1, 4), (-1, 2), (q, 1), (q^3, 1), (t, 2), (t^2, 2)\}$$

Here $q$ is a primitive fourth root of unity, and $t$ is a primitive cube root of unity.

In fact, one can read the cycle type of a permutation directly from its eigenvalues. The algorithm is as follows. Find the largest $m$ such that an $m$th root of unity appears in the list of eigenvalues with nonzero multiplicity. Suppose it appears with multiplicity $k$. Your permutation has $k$ cycles of length $m$. Now let $q$ be a primitive $m$-th root of unity and remove $k$ copies of each of $\{q, q^2, \ldots, q^m\}$ from the list of eigenvalues. Repeat.

Since two permutations cannot be conjugate in $GL(n)$, let alone in in $\mathfrak{S}_n$ unless they have the same eigenvalues, we see that having the same cycle type is also a *necessary* condition for two permutations to be conjugate.