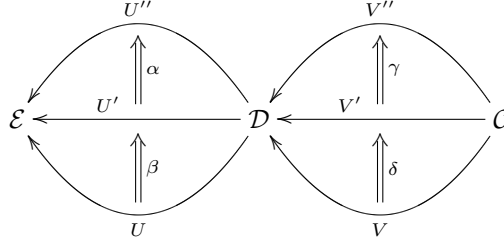


Suppose that we have categories, functors, and natural transformations as suggested by the following diagram:



It's not yet clear at this point, even though we have horizontal and vertical composition, that this diagram defines a well-defined composite, since there are at least two ways we could put this a diagram together. We could either take the vertical composites $\alpha \circ \beta$ and $\gamma \circ \delta$, and then horizontally compose them, or we could take the horizontal composites $\alpha \bullet \gamma$ and $\beta \bullet \delta$ and then vertically compose them. However:

$$\begin{aligned}
 & (\alpha \circ \beta) \bullet (\gamma \circ \delta) \\
 = & (\alpha \circ \beta)V'' \circ U(\gamma \circ \delta) && \text{def. of } \bullet \\
 = & \alpha V'' \circ \beta V'' \circ U\gamma \circ U\delta && \text{distrib.} \\
 = & \alpha V'' \circ (\beta \bullet \gamma) \circ U\delta && \text{def. of } \bullet \\
 = & \alpha V'' \circ (U'\gamma \circ \beta V') \circ U\delta && \text{equiv. def. of } \bullet \\
 = & (\alpha \bullet \gamma) \circ (\beta \bullet \delta) && \text{def. of } \bullet
 \end{aligned}$$

(Note that we used the associativity of \circ a bunch here, but I left those steps out.)
So indeed, we have the law that:

$$\boxed{(\alpha \circ \beta) \bullet (\gamma \circ \delta) = (\alpha \bullet \gamma) \circ (\beta \bullet \delta)}$$

which is known as the *interchange law*.

The interchange law and the associativity of vertical and horizontal composition together give us the somewhat-awkward-to-formalise but intuitive result that if we have any diagram of natural transformations that we could compose, it doesn't matter what order we choose to do it in. So the pasting diagrams that we've been using, being somewhat ambiguous about it, are really an appropriate notation to use.

It's also possibly worth noting here that the functor distributivity laws we used to prove the interchange law and the associativity of horizontal composition are also special cases of the interchange law. For example:

$$\begin{aligned}
 & (\alpha \circ \beta)V \\
 = & (\alpha \circ \beta) \bullet \text{id}_V \\
 = & (\alpha \circ \beta) \bullet (\text{id}_V \circ \text{id}_V) \\
 = & (\alpha \bullet \text{id}_V) \circ (\beta \bullet \text{id}_V) \\
 = & \alpha V \circ \beta V
 \end{aligned}$$

There's a valuable analogy to be had here with the multiplication and tensor product of linear maps.

If we have vector spaces V, V', W, W' , and linear maps $A : V \rightarrow V'$ and $B : W \rightarrow W'$ then we can define the tensor product of linear maps:

$$\begin{aligned} A \otimes B &: V \otimes W \rightarrow V' \otimes W' \\ (A \otimes B)(v \otimes w) &= (Av) \otimes (Bw) \end{aligned}$$

With this definition, we have that:

$$\begin{aligned} &((A \circ B) \otimes (C \circ D))(v \otimes w) \\ &= ((A \circ B)v \otimes (C \circ D)w) \\ &= (A(Bv) \otimes C(Dw)) \\ &= (A \otimes C)((Bv) \otimes (Dw)) \\ &= (A \otimes C)((B \otimes D)(v \otimes w)) \\ &= ((A \otimes C) \circ (B \otimes D))(v \otimes w) \end{aligned}$$

so we have that:

$$(A \circ B) \otimes (C \circ D) = (A \otimes C) \circ (B \otimes D)$$

The situation for natural transformations, vertical, and horizontal composition looks so similar to the one for linear maps, composition, and tensor product that it seems there must be something going on here.

In fact there is, and in order to explain it, we'll need the notion of a weak 2-category, which I'll save for another time.