Okay, so we've seen at least a handful of examples of these natural transformation things. They are actually all over the place if you know where to look for them. As we move along through other categorical structures, we'll see a fair number of them.

So, if natural transformations are supposed to be the arrows between functors, and we're intending to do some category theory, we should be unsatisfied with just an unstructured set of them. We really ought to be able to compose them! Owing to their two-dimensional nature, there will actually be two ways.

Let's start with the most obvious one, the one which turns the collection of functors between any two categories into a category itself.

Suppose we have categories, functors and natural transformations as depicted in this diagram:

$$\begin{tikzcd}[column sep=huge]
\mathcal{D} & \mathcal{C} \arrow[l, "G"', ""{name=G}] \arrow[l, bend right=60, "H"', ""{name=H}] \arrow[l, bend left=60, "F", ""'{name=F}] \arrow[from=F, to=G, Rightarrow, "\eta"'] \arrow[from=G, to=H, Rightarrow, "\lambda"']
\end{tikzcd}$$

So, these line up nicely, and for any object $X \in \mathcal{C}$ we have components:

$$\begin{tikzcd}
HX \\
GX \arrow[u, "\lambda_X"'] \\
FX \arrow[u, "\eta_X"']
\end{tikzcd}$$

and if we have any arrow in $\mathcal{C}$:

$$\begin{tikzcd}
Y & X \arrow[l, "a"']
\end{tikzcd}$$

then we have the following diagram in $\mathcal{D}$:

$$\begin{tikzcd}
HY & HX \arrow[l, "Ha"'] \\
GY \arrow[u, "\lambda_Y"'] & GX \arrow[l, "Ga"'] \arrow[u, "\lambda_X"'] \\
FY \arrow[u, "\eta_Y"'] & FX \arrow[l, "Fa"'] \arrow[u, "\eta_X"']
\end{tikzcd}$$

Since each naturality square commutes, the outer rectangle does as well:

$$\begin{aligned}
&\lambda_Y \circ \eta_Y \circ Fa \\
=&\lambda_Y \circ Ga \circ \eta_X \\
=&Ha \circ \lambda_X \circ \eta_X
\end{aligned}$$

which means that we have a new natural transformation, called the *vertical composite* of λ and η defined by:

$$\boxed{(\lambda \circ \eta)_X = \lambda_X \circ \eta_X}$$

(The above equation is in a box, because it is so very important to remember when working with these things!)

Moreover, we have identity and associativity fall out essentially immediately from the identity and associativity of arrows in $\mathcal{D}$. (If this isn't clear, then check it!)

That is, for any natural transformation $\alpha : U \to U'$,

$$\boxed{\begin{aligned} \mathrm{id}_{U'} \circ \alpha &= \alpha \\ \alpha \circ \mathrm{id}_U &= \alpha \end{aligned}}$$

where id_U is the identity natural transformation:

$$\boxed{(\mathrm{id}_U)_X = \mathrm{id}_{(UX)}}$$

and if we have natural transformations like this:

$$\begin{tikzcd}[column sep=huge]
\mathcal{D} & \mathcal{C} \arrow[l, bend right=70, ""{name=T, below}] \arrow[l, bend right=20, ""{name=M1, above}, ""{name=M1b, below}] \arrow[l, bend left=20, ""{name=M2, above}, ""{name=M2b, below}] \arrow[l, bend left=70, ""{name=B, above}]
\arrow[Rightarrow, from=B, to=M2b, "\alpha"']
\arrow[Rightarrow, from=M2, to=M1b, "\beta"']
\arrow[Rightarrow, from=M1, to=T, "\gamma"']
\end{tikzcd}$$

then,

$$\boxed{(\alpha \circ \beta) \circ \gamma = \alpha \circ (\beta \circ \gamma)}$$

Now let's turn to the second type of composition. This time, we have categories and functors like so:

$$\begin{tikzcd}[column sep=huge]
\mathcal{E} & \mathcal{D} \arrow[l, bend right=50, "U'"', ""{name=U1, below}] \arrow[l, bend left=50, "U", ""{name=U0, above}] & \mathcal{C} \arrow[l, bend right=50, "V'"', ""{name=V1, below}] \arrow[l, bend left=50, "V", ""{name=V0, above}]
\arrow[Rightarrow, from=U0, to=U1, "\alpha"']
\arrow[Rightarrow, from=V0, to=V1, "\beta"']
\end{tikzcd}$$

So, for any $X \in \mathcal{C}$ we have the following components (examine this from right to left, as I've aligned the bits here with which category they lie in, in the diagram above):

$$\begin{tikzcd}
U'V'X & U'VX \arrow[l, "U'\beta_X"'] & V'X & \\
 & & & X \\
UV'X \arrow[uu, "\alpha_{V'X}"'] & UVX \arrow[l, "U\beta_X"'] \arrow[uu, "\alpha_{VX}"'] & VX \arrow[uu, "\beta_X"'] &
\end{tikzcd}$$

Because α is natural, we have that the square in $\mathcal{E}$ actually commutes:

$$\alpha_{V'X} \circ U\beta_X = U'\beta_X \circ \alpha_{VX}$$

So, we will define the *horizontal composite* of this α and β as the common value:

$$\boxed{(\alpha \bullet \beta)_X = \alpha_{V'X} \circ U\beta_X = U'\beta_X \circ \alpha_{VX}}$$

This might seem complicated and hard to remember, but we can simplify it a bit if we define (for any appropriate α and V):

$$\boxed{(\alpha V)_X = \alpha_{(VX)}}$$

so that if

$$\alpha : U \to U'$$

then

$$\alpha V : UV \to U'V$$

And similarly,

$$\boxed{(U\beta)_X = U(\beta_X)}$$

so that if

$$\beta : V \to V'$$

then

$$U\beta : UV \to UV'$$

Then the definition above becomes

$$\boxed{(\alpha \bullet \beta)_X = (\alpha V' \circ U\beta)_X = (U'\beta \circ \alpha V)_X}$$

or more abstractly,

$$\boxed{\alpha \bullet \beta = \alpha V' \circ U\beta = U'\beta \circ \alpha V}$$

The notation is starting to better reflect the notion that we're just doing α on the U part, and β on the V part, and it doesn't matter which one we do first, so long as we do them both.

The notation is also getting simpler. As we develop the properties of vertical and horizontal composition, this will all get conceptually much easier to keep track of.

One thing which is nice to notice is that these new notations we've defined for sticking a functor onto a natural transformation are the same as horizontal composition with the identity natural transformation on that functor:

$$\begin{aligned}
(\alpha \bullet \mathrm{id}_V)_X &= U'(\mathrm{id}_V)_X \circ \alpha_{VX} && \text{def. of } \bullet \\
&= U'\mathrm{id}_{VX} \circ \alpha_{VX} && \text{component of identity} \\
&= \mathrm{id}_{U'VX} \circ \alpha_{VX} && U' \text{ is a functor} \\
&= \alpha_{VX} && \text{identity} \\
&= (\alpha V)_X && \text{notation}
\end{aligned}$$

The proof that $\mathrm{id}_U \bullet \beta = U\beta$ is similar, and a good exercise in checking that you're comfortable with the definitions and notation.

We also have that $\bullet$ is associative. I'm going to start working entirely at the level of functors and natural transformations here, since writing X in so many places is becoming tedious, and starts to make the presentation less clear. You're invited to check anything which is not immediately apparent (in particular, the dubious steps that I've marked with (!?)).

We have functors and natural transformations as follows:

$$\begin{tikzcd}
\mathcal{F} & & \mathcal{E} \arrow[ll, bend right=50, "U'"'] \arrow[ll, bend left=50, "U"] \arrow[ll, phantom, "\Uparrow\alpha"] & & \mathcal{D} \arrow[ll, bend right=50, "V'"'] \arrow[ll, bend left=50, "V"] \arrow[ll, phantom, "\Uparrow\beta"] & & \mathcal{C} \arrow[ll, bend right=50, "W'"'] \arrow[ll, bend left=50, "W"] \arrow[ll, phantom, "\Uparrow\gamma"]
\end{tikzcd}$$

$$\begin{aligned}
(\alpha \bullet \beta) \bullet \gamma &= (\alpha \bullet \beta)W' \circ UV\gamma && \text{def. of } \bullet \\
&= (\alpha V' \circ U\beta)W' \circ UV\gamma && \text{def. of } \bullet \\
&= (\alpha V'W' \circ (U\beta)W') \circ UV\gamma && \text{distrib. (!?)} \\
&= \alpha V'W' \circ ((U\beta)W' \circ UV\gamma) && \text{assoc. of } \circ \\
&= \alpha V'W' \circ (U(\beta W') \circ UV\gamma) && \text{assoc. (!?)} \\
&= \alpha V'W' \circ U(\beta W' \circ V\gamma) && \text{distrib. (!?)} \\
&= \alpha V'W' \circ U(\beta \bullet \gamma) && \text{def. of } \bullet \\
&= \alpha \bullet (\beta \bullet \gamma) && \text{def. of } \bullet
\end{aligned}$$

That's obviously important so let's put it in a box:

$$\boxed{(\alpha \bullet \beta) \bullet \gamma = \alpha \bullet (\beta \bullet \gamma)}$$

It's slowly becoming more apparent that perhaps it doesn't matter at all about which order we choose to paste some complicated diagram of natural transformations together, and that everything is perfectly associative in every possible way that it could be.

We almost have it, there is but one remaining basic property we'll need, and I'll save it for the next post.