

Let's have a look at the details of just a couple more examples of natural transformations, this time from linear algebra.

One example is just so standard that I can't help but give it. If we have some finite dimensional vector space V over the field F , then the dual space V^* is the space of all linear transformations $V \rightarrow F$, which in this special case are referred to as linear functionals. It's not too hard to show that V^* has the same dimension as V , and therefore must be isomorphic to it. One way is simply to note that the dimension of the space of linear maps $V \rightarrow W$ is equal to the product of the dimensions of V and W , and that the dimension of F over itself is of course 1.

However, there's no obvious isomorphism to pick. If you fix a basis, there's a sort of obvious one to choose from there, but each possible choice of basis for V leads to a different isomorphism.

However, consider the double dual: V^{**} which is the space of linear maps $V^* \rightarrow F$. The situation suddenly seems much different. Of course, again we have that V is isomorphic to V^{**} , but this time there is a natural choice of isomorphism $V \rightarrow V^{**}$, in particular,

$$\begin{aligned} \eta_V : V &\rightarrow V^{**} \\ v &\mapsto (f \mapsto f(v)) \end{aligned}$$

That is, we send a vector v to the linear map which takes a linear functional f and sends it to $f(v)$.

So, there's a sense here that this choice of isomorphism seems somehow "natural", and it's certainly quite special, but how do we formalise that?

Well, we'd like to be able to say that this double dual thing was a functor, and then we could apply our categorical definition of what natural means.

So let's look at the dual first, and consider a linear map $T : V \rightarrow W$. Can we construct from it a linear map $V^* \rightarrow W^*$? Well, it seems not in any obvious way aside from choosing particular isomorphisms.

How about $W^* \rightarrow V^*$ then? We could go for making $*$ into a functor:

$$* : \mathbf{Vec} \rightarrow \mathbf{Vec}^{\text{op}}.$$

Given a linear functional $g : W \rightarrow F$ and a linear map $T : V \rightarrow W$, we can just compose them to get a linear functional $g \circ T : V \rightarrow F$.

This process of converting linear maps $V \rightarrow W$ into linear maps $W^* \rightarrow V^*$ is usually called the transpose (as it corresponds to taking the transpose of matrix representations in a particular way), and we define

$$T^t(g) = g \circ T.$$

We also have that

$$\begin{aligned}
 (T \circ U)^t(g) &= g \circ (T \circ U) \\
 &= (g \circ T) \circ U \\
 &= T^t(g) \circ U \\
 &= U^t(T^t(g)) \\
 &= (U^t \circ T^t)(g)
 \end{aligned}$$

and of course this will send the identity transformation to the identity transformation. So this really is an honest functor.

So, this means that the map $**$ which sends each vector space to its double dual and each linear map to the transpose of its transpose is now a plain functor

$$** : \mathbf{Vec} \rightarrow \mathbf{Vec}.$$

Our η_V from above is now primed to become the component of a natural transformation:

$$\eta : \text{id}_{\mathbf{Vec}} \rightarrow **$$

All we have to check is that this diagram commutes:

$$\begin{array}{ccc}
 V & \xrightarrow{T} & W \\
 \eta_V \downarrow & & \downarrow \eta_W \\
 V^{**} & \xrightarrow{T^{tt}} & W^{**}
 \end{array}$$

Easy enough, there's lots of flipping things around and notation, but it's all mechanical. For any $v \in V$ and $g \in W^*$, we have:

$$\begin{aligned}
 (\eta_W \circ T)(v)(g) &= \eta_W(Tv)(g) && \text{composite} \\
 &= g(Tv) && \text{def. of } \eta_W \\
 (T^{tt} \circ \eta_V)(v)(g) &= T^{tt}(\eta_V(v))(g) && \text{composite} \\
 &= (\eta_V(v) \circ T^t)(g) && \text{def. of transpose} \\
 &= \eta_V(v)(T^t(g)) && \text{composite} \\
 &= T^t(g)(v) && \text{def. of } \eta_V \\
 &= (g \circ T)(v) && \text{def. of transpose} \\
 &= g(Tv) && \text{composite}
 \end{aligned}$$

So it works out, and this really is a natural transformation.