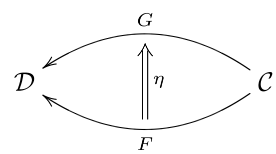
I'd like to let a larger group of people in on my series of posts about adjunctions and monads, which itself is designed to let a larger group of people in on this stuff I'd like to say about vector spaces, functional programming, and other areas.

To that end, I'll push the stuff I'm going to write about adjunctions onto the stack, and do a few posts right now about natural transformations. You should hopefully be able to follow from here if you already are comfortable with the basic definitions of what categories and functors are, but I will be throwing some other topics into the examples for enrichment.

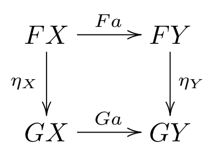
So, to start things off, let's have a look at what natural transformations are and have a look at a couple examples before we get into what things we can do with them.

Suppose that we have categories $\mathcal{C}$ and $\mathcal{D}$ and two parallel functors. Our natural transformation $\eta : F \to G$ will go between the functors:



(The reason our arrows in diagrams of natural transformations will go from right to left and and bottom to top is that it will make composition formulas easier to read off later.)

The natural transformation consists of, for each object $X \in \mathcal{C}$, an arrow $\eta_X : FX \to GX$ in $\mathcal{D}$. Furthermore, for every arrow $a : X \to Y$ in $\mathcal{C}$, we have the following commutative square in $\mathcal{D}$:



That is, $Ga \circ \eta_X = \eta_Y \circ Fa$.

A good interpretation of this naturality square in many settings is that if we're thinking of each FX as some sort of structure built on X in some way, and Fa as acting as a does on the bits which came from X and leaving the additional structure alone, then the component of the natural transformation η_X acts on the other half: it rearranges the F structure into a G structure, leaving the bits which come from X alone.

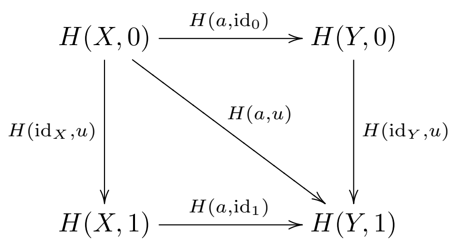
There is an alternate definition from which we might be able to extract some more intuition that I would like to take a quick look at here:

We will define the categorical interval $\mathbf{I}$ as the extremely simple category with two objects $\{0, 1\}$, and a unique nonidentity arrow $u : 0 \to 1$ between them. (This category is also sometimes known as $\mathbf{2}$.)

Suppose that F and G are functors as before. Then, a *homotopy of functors* $F \to G$ is defined to be a functor $H : \mathcal{C} \times \mathbf{I} \to \mathcal{D}$ such that:

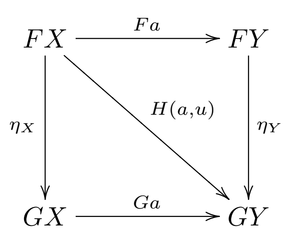
$$\begin{aligned} H(X, 0) &= FX \\ H(X, 1) &= GX \\ H((a, \mathrm{id}_0) : (X, 0) \to (Y, 0)) &= Fa \\ H((a, \mathrm{id}_1) : (X, 1) \to (Y, 1)) &= Ga \end{aligned}$$

If we have a functor like this, then for any $a : X \to Y$ in $\mathcal{C}$, we have the following diagram of arrows in $\mathcal{D}$:



Because the composite of arrows in the product category is defined by $(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ f_2, g_1 \circ g_2)$, we have that each of those two triangles commutes, and so the square does as well.

But now if we can define for each $X \in \mathcal{C}$, the arrow $\eta_X = H(\mathrm{id}_X, u)$ we can replace things in the above diagram according to the equations we have to get:



So these components η_X give rise to a natural transformation $\eta : F \to G$.

Moreover, if we start with a natural transformation, $\eta : F \to G$, we can define a homotopy of functors using $H(\mathrm{id}_X, u) = \eta_X$, and construct the remainder of the definition of H according to the laws it must satisfy. So the two definitions are equivalent.

Those familiar with topology at this point are probably quite happy with this, but for everyone else who is in the dark, the analogy here is to the concept of a homotopy of continuous maps.

In topology, we might have topological spaces $\mathcal{C}$ and $\mathcal{D}$ (if you don't know about topological spaces, you can think of subsets of $\mathbb{R}^n$ perhaps), and a pair of continuous functions $F, G : \mathcal{C} \to \mathcal{D}$. The notion we're trying to define is when it's possible to continuously deform the function F into the function G. We define a homotopy of continuous maps as a continuous map $H : \mathcal{C} \times \mathbf{I} \to \mathcal{D}$, where this time, our $\mathbf{I}$ is the interval of real numbers:

$$\mathbf{I} = [0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$$

subject to the conditions that for each point $X \in \mathcal{C}$:

$$H(X, 0) = F(X)$$
$$H(X, 1) = G(X)$$

The way we typically think of this is to consider the second parameter $t \in [0, 1]$ to be like time. If we pick some point $X \in \mathcal{C}$, then the map $H(X, -) : [0, 1] \to \mathcal{D}$ will trace out a continuous curve in $\mathcal{D}$ which goes from $F(X)$ at $t = 0$ to $G(X)$ at $t = 1$. This curve is analogous to our component $\eta_X : FX \to GX$.

It seems no accident that the founders of category theory were topologists.