

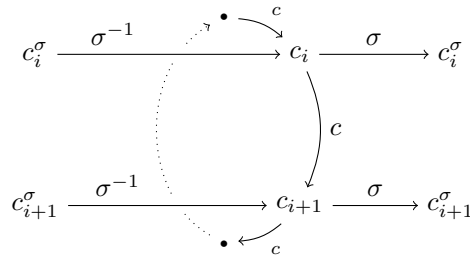
I had a feeling that while the use of linear algebra in Robin's notes about the action of conjugation in permutation groups is interesting, I thought it would be nice to see why the conjugacy classes are determined by cycle type by a purely combinatorial argument. After a chat about it, Robin encouraged me to write this note.

Let's start by looking just at what happens when we conjugate a single cycle $c = (c_1, c_2, \dots, c_k)$ by an arbitrary permutation σ , both in \mathfrak{S}_n , giving:

$$c^\sigma = (c_1, c_2, \dots, c_k)^\sigma = \sigma^{-1}(c_1, c_2, \dots, c_k)\sigma$$

First of all, which elements of $[n] = \{1, 2, \dots, n\}$ will be affected?

Diagrammatically, we have something which looks like this:



Here, the vertices are elements of $[n]$, and the arrows show where various permutations send them, according to which permutation they are labelled with.

There are two sets of elements to consider: those which are sent to some c_i by σ^{-1} , and so are equal to c_i^σ for some i , and those which are not.

If x is not sent to something affected by the cycle by σ^{-1} , so that

$$(x^{\sigma^{-1}})^c = x^{\sigma^{-1}}$$

then we have:

$$x^{\sigma^{-1}c\sigma} = (x^{\sigma^{-1}})^{c\sigma} = (x^{\sigma^{-1}})^\sigma = x$$

so that x is unaffected by the conjugated cycle.

On the other hand, if

$$x = c_i^\sigma$$

we have that:

$$(c_i^\sigma)^{\sigma^{-1}c\sigma} = c_i^{c\sigma} = c_{i+1}^\sigma$$

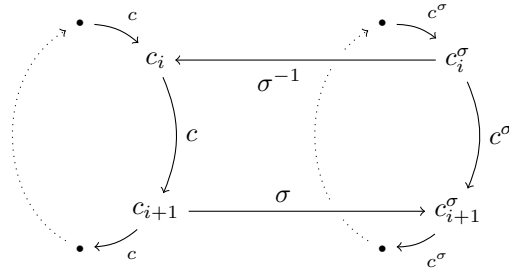
(You can follow this in the diagram above, by starting at the c_i^σ and taking the σ^{-1} , c , and σ arrows successively.)

So the upshot of this is that:

$$(c_1, \dots, c_k)^\sigma = (c_1^\sigma, \dots, c_k^\sigma)$$

As each c_i^σ is sent by c^σ to c_{i+1}^σ , and everything else is left untouched.

To rearrange the diagram a little to reflect this new insight, we have something like this:



The trivial but important corollary here is that the image of a cycle under conjugation is always another cycle of the same length.

Another important result is that given two cycles of the same length, say $c = (c_1, \dots, c_k)$ and $d = (d_1, \dots, d_k)$ we can easily find a permutation σ , conjugation by which will send one to the other. Start with a function

$$\begin{aligned} \sigma^* : \{c_1, \dots, c_k\} &\rightarrow \{d_1, \dots, d_k\} \\ c_i &\mapsto d_i \end{aligned}$$

This will be well-defined, since each element of the first cycle occurs only once, and injective since any element will only occur once in the second cycle. So it will extend to a permutation in \mathfrak{S}_n (in a number of different ways in general). Any such permutation will do.

Since there are k different ways to line up the elements of the two cycles, and $(n - k)!$ ways to rearrange the remainder of the elements, there will be $k(n - k)!$ permutations that conjugate any single cycle of length k in \mathfrak{S}_n to any other cycle of length k .

So individual cycles are conjugate if and only if they have the same length.

The next thing to note is that since every permutation can be written as a product of disjoint cycles, and conjugation is a group homomorphism, we have that conjugation preserves cycle type.

But not only does it preserve cycle type, as with individual cycles, given two permutations having the same cycle type, we can easily construct a permutation that sends one to the other, in much the same way as we did with the individual cycles. Just line up all the cycles of the same length in each permutation (there'll be some choice in the matter, but any choice will do), and define a permutation sending each element of each cycle to the corresponding element of the corresponding cycle. Of course, unlike the case with just a single cycle, after choosing how to line things up this time, there will be no remaining choices to make (provided that we include cycles of length 1 in the process, if any).

For example, if we have:

$$\begin{aligned} a &= (1, 2)(3, 6, 7)(5, 9, 12)(4, 8, 11, 10) \\ b &= (5, 8)(1, 3, 6)(2, 10, 4)(7, 12, 11, 9) \end{aligned}$$

then we can simply read off the columns, choosing

$$1^\sigma = 5, 2^\sigma = 8, 3^\sigma = 1, \dots$$

to get

$$\sigma = (1, 5, 2, 8, 12, 4, 7, 6, 3)(9, 10)(11)$$

and then $a^\sigma = b$.

Rather than checking that we did this correctly by hand, we can confirm it using `gap`, as follows:

```
gap> a := (1,2)(3,6,7)(5,9,12)(4,8,11,10);
(1,2)(3,6,7)(4,8,11,10)(5,9,12)
gap> b := (5,8)(1,3,6)(2,10,4)(7,12,11,9);
(1,3,6)(2,10,4)(5,8)(7,12,11,9)
gap> sigma := (1,5,2,8,12,4,7,6,3)(9,10)(11);
(1,5,2,8,12,4,7,6,3)(9,10)
gap> a^sigma = b;
true
```

And there you have it.

To summarise, conjugacy of permutations is entirely determined by their cycle type, and in general, there will be one permutation that conjugation by which sends one to the other for each way to align corresponding cycles of the same length.

So, if the cycle type is some partition:

$$\lambda = (\lambda_1^{m_1}, \dots, \lambda_k^{m_k})$$

where the λ_i are distinct, and the m_i indicate the number of times that cycle length occurs (and so are not an exponent in the usual sense), then since we have $m_i!$ ways to pair up cycles of length λ_i , and then for each of these cycles, we can choose from λ_i cyclic shifts to pair up members of that cycle, we have that the total number of permutations sending one member of this conjugacy class to another under conjugation are:

$$z_\lambda = \prod_{i=1}^k m_i! \lambda_i^{m_i}$$

For example, my permutations above both had cycle type $(2, 3^2, 4)$, and so the number of ways we could have constructed σ , by rearranging the cycles in b is:

$$z_{(2,3^2,4)} = 2 \cdot 2! 3^2 \cdot 4 = 144$$

Because of this computation, we can tell how many elements are in each conjugacy class. If for every permutation that sends a to b , we have exactly z_λ of them which do the same thing, we have that starting from any permutation a , there are $\frac{n!}{z_\lambda}$ distinct things we can get by conjugating with each of the $n!$ permutations in \mathfrak{S}_n , which is then the size of the conjugacy class \mathfrak{S}_λ .