Suppose we have almost any deduction system whatsoever. (I'm going to refine things and ask for more structure as we go along, but for now it will do.)

Our deduction system consists of a bunch of statements, and some rules for determining when some statements follow from others. It's not much of a stretch to demand that given any statement X, we will be able to prove X from it, that is, that we have some trivial deduction $X \vdash X$. Also, in order to keep things interesting, if we have a deduction $X \vdash Y$, and a deduction $Y \vdash Z$, we ought to have some way to join them into a deduction $X \vdash Z$. This will be some sort of ”concatenation of proofs”, but we're not going to care how it's implemented, just so long as it is associative, and the ”identity deduction” above is an identity for it.

So, anyone who knows me well enough should know what's coming now. We form a category whose objects are statements, and where the set of arrows $X \to Y$ are exactly the deductions $X \vdash Y$. That is, the proofs of Y given X. The identity arrows are given by trivial deductions, and the composition is our concatenation of proofs.

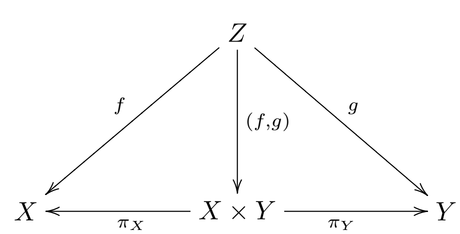
Let's rediscover basic logical operations in terms of category theoretic constructions.

Starting simple, what would an initial object 0 correspond to? Well, the property of an initial object is that for any X we have a (unique) arrow $0 \to X$. This seems to suggest that 0 ought to be a statement which is trivially false, as anything can be deduced from it.

How about a terminal object 1? There we have for any X, a unique arrow $X \to 1$. So this is behaving like something trivially true.

You might recall that in an arbitrary category with a terminal object, we often define an ¡em¿element¡/em¿ of an object X to be an arrow $1 \to X$. What would that be here? Well, it's a proof of X starting from something which is trivially true, or just a proof of X. Dually, a coelement of X which is an arrow $X \to 0$ will be a proof of falsity from X, or a refutation of X.

How about the product of statements X and Y? Well, $X \times Y$ should be a statement which, according to the projection maps, implies both X and Y, and moreover, if any other statement Z implies both X and Y, then there must be a (unique) proof that Z implies $X \times Y$, making the following diagram commute:



These are the usual properties attributed to logical AND, so the existence of (finite) products is, modulo perhaps some squabbling about uniqueness of proofs, equivalent to the notion that we have statements representing the conjunctions of other statements.

In a similar fashion, the coproduct $X + Y$ has essentially all the properties we'd expect from logical disjunction (OR) of statements.

How about exponential objects? Recall that if X and Y are objects in some category $\mathcal{C}$, then an exponential object Y^X is an object of $\mathcal{C}$ together with an arrow

$$\text{apply} : Y^X \times X \to Y$$

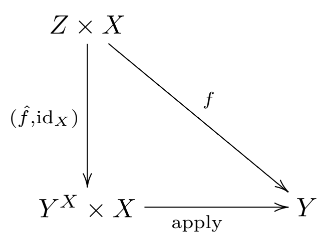
so that whenever there is an arrow

$$f : Z \times X \to Y$$

we have that there is a unique arrow

$$\hat{f} : Z \to Y^X$$

such that the following diagram commutes:



In order to better follow this, we can try setting Z to be 1, in which case f becomes simply an arrow $X \to Y$, and $\hat{f}$ becomes an arrow $1 \to Y^X$, which we can think of as an element of the exponential.

That is, every proof of Y^X should correspond to a proof that $X \to Y$.

So exponential objects correspond directly with implication.

What about logical negation? Well, we said earlier that an arrow $X \to 0$ would act like a refutation of X, so a natural thing to do would be to define $\neg X$ to be 0^X.

Okay, so if our deduction system is in fact a Cartesian closed category with coproducts, it means we have encodings of the usual logical operations. What about the usual properties of logical operations with respect to each other?

Firstly, can we prove the distributive laws? In classical logic, we have that

$$A \times (B + C) \iff (A \times B) + (A \times C)$$

and that

$$A + (B \times C) \iff (A + B) \times (A + C)$$

Secondly, do we get De Morgan's laws for free, or do we need to add something more to this formulation?

I'll leave those questions to be answered next time.